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Stability of Finite Difference Schemes for the Problem of Elastic Wave Propagation in a Quarter Plane

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For the problem of elastic waves propagating in a quarter plane, the stability of the finite difference method is critically dependent on the approximation of the boundary conditions and the treatment of the 90° corner. The stability of two classical and two composed approximations to the boundary conditions is studied using analysis of the local propagating matrix and by computer experiments. Mistreatment of certain grid points near the corner is found to be the cause of the unstable solution reported by Alterman and Rotenberg (1969). Correction of this results in a stable scheme in which the range of stability of the different approximations of boundary conditions for the quarter plane is the same as for the half plane. The two classical approximations, which use fictitious lines of grid points, are reliable for quarter planes only when the ratio of shear velocity to compressional velocity $\beta/\alpha > 0.5$. For $\beta/\alpha < 0.5$ the results they give are contaminated, by an oscillation about the true solution in the case of the central difference approximation and also by a phase delay in the one-sided approximation. The first composed approximation (A. Ilan *et al.*, *Geophys. J. R. Astron. Soc.* **34** (1975), 727–742) is stable only for $\beta/\alpha >$ 0.575. However, the new composed approximation (A. Ilan and D. Loewenthal, *Geophys. Prospect.* **24** (1976), 431–453) is shown to be stable even for small values of β/α.

1. INTRODUCTION

The problem of elastic wave propagation in media containing sharp corners, steps, and cracks is of practical interest for seismology, nondestructive testing, and other engineering fields. This problem has been investigated both experimentally and through mathematical methods. The finite difference method has proved itself to be a useful tool for investigation of wave propagation. It was applied to seismological problems by Alterman and Karal [1] and has since been extensively used and developed.

Two of the most delicate tasks in finite difference methods are the introduction of boundary conditions into the scheme and the treatment of singular points, such as corners and tips of cracks. The manner in which these tasks are carried out may have a critical influence on the accuracy and stability of results. There are cases in which a scheme gives accurate results for a certain range of parameters but is inaccurate and even unstable for parameters outside this range. Alterman and Rotenberg [2] reported

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on cases in which a scheme is stable for a half-plane but 'explodes' numerically when applied to a quarter plane.

The aim of the present paper is to investigate the accuracy and stability of different approximations to boundary conditions in a quarter-plane. Ilan and Loewenthal [8] studied the problem of instability due to boundary conditions in a half-plane. In a quarter-plane, the interaction between schemes for the two perpendicular surfaces as well as the treatment of the corner introduces further difficulties and may cause more restrictions on the parameters of the model. Since a quarter-plane is a basic element of geometries containing discontinuities, the range of applicability of each approximation is obviously of interest.

The theory of stability of finite difference schemes with initial and boundary conditions is not as developed as the theory for pure initial value problems. In the last decade an important development has been made in this field. A theory of stability of the mixed initial and boundary value problems was proposed by Godunov and Ryabenkii [5], modified and developed by Kreiss [10], and applied to various cases notably by Osher [11] and Gustafsson *et al.* [6]. This theory was applied mostly to geometries consisting of a segment of one-dimensional line. It can be generalized to be applied also on a half-plane or a strip, but apparently it is not applicable to media containing corners.

Ilan and Loewenthal [8] proposed a technique for checking the stability of a given finite difference scheme, which takes into account the approximation of the boundary conditions. It is based on analyzing a propagating matrix associated with a small sample of the grid including surface points. In the present paper this method is applied to a quarter-plane. Considering the matrix of propagation enables us to point out the causes of instability of cases reported by Alterman and Rotenberg [2], and to propose a remedy for the situation. The paper is organized as follows: Section 2 describes the model. Section 3 includes the finite difference scheme with four different approximations to boundary conditions and two treatments of the corenr. The results of the different approximations are compared in Section 4. The causes of instability of the central approximation applied to a quarter-plane are detected and the scheme is changed slightly to make it stable. Section 4 includes also numerical experiments for determination of the range of stability and of the applicability of each approximation. In Section 5 the matrix of propagation associated with a sample of typical grid in a quarter plane is analyzed.

2. Model Assumptions

In order to study instability due to boundary conditions in media containing corners the simplest geometry of a quarter-plane, has been chosen. The material is assumed to be perfectly elastic, isotropic, and homogeneous, α is the compressional and β is the shear velocity. The quarter-plane is excited by a compressional point source at the origin of the Cartesian coordinate system. Let the x axis be parallel to the horizontal free surface with z pointing vertically upwards. Let (U, W) be the horizontal and vertical displacements, respectively, and t denote the time. The equations of motion are as follows:

$$\frac{\partial^2 U}{\partial t^2} = \alpha^2 \frac{\partial^2 U}{\partial x^2} + (\alpha^2 - \beta^2) \frac{\partial^2 W}{\partial x\,\partial z} + \beta^2 \frac{\partial^2 U}{\partial z^2}$$

$$\frac{\partial^2 W}{\partial t^2} = \alpha^2 \frac{\partial^2 W}{\partial z^2} + (\alpha^2 - \beta^2) \frac{\partial^2 U}{\partial x\,\partial z} + \beta^2 \frac{\partial^2 W}{\partial x^2}. \tag{1}$$

The initial conditions are

$$U = W = \frac{\partial U}{\partial t} = \frac{\partial W}{\partial t} = 0; \qquad t \leqslant 0. \tag{2}$$

At time $t = 0$ the point source starts to emit a compressional pulse. The radial displacement $S(r, t)$ due to the source at a distance $r = (x^2 + z^2)^{1/2}$ is given by

$$S(r, t) = \frac{1}{\Delta^4} [G(r, t) - 4G(r, t - \Delta) + 6G(r, t - 2\Delta)$$
$$- 4G(r, t - 3\Delta) + G(r, t - 4\Delta)],$$

where

$$G(r, t) = \frac{ar^3}{96\alpha^4} [(q^2 - 1)^{1/2} (2q^3 + 13q)$$
$$- 3(4q^2 + 1) \log((q^2 - 1)^{1/2} + q)]\, H(q - 1). \tag{3}$$

Here $q = t\alpha/r$, a is a constant, H is the Heaviside's unit step function, and Δ is a parameter which determines the width of the pulse. $S(r, t)$ is a source function which was found to be suitable for finite difference schemes due to its high order of smoothness in [8].

The boundary conditions of the stress free surface are

$$\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = 0$$
$$(\alpha^2 - 2\beta^2) \frac{\partial U}{\partial x} + \alpha^2 \frac{\partial W}{\partial z} = 0 \qquad \text{on } z = \text{constant for all } t \tag{4}$$

$$\frac{\partial U}{\partial z} + \frac{\partial W}{\partial z} = 0$$
$$(\alpha^2 - 2\beta^2) \frac{\partial W}{\partial z} + \alpha^2 \frac{\partial U}{\partial x} = 0 \qquad \text{on } x = \text{constant for all } t. \tag{5}$$

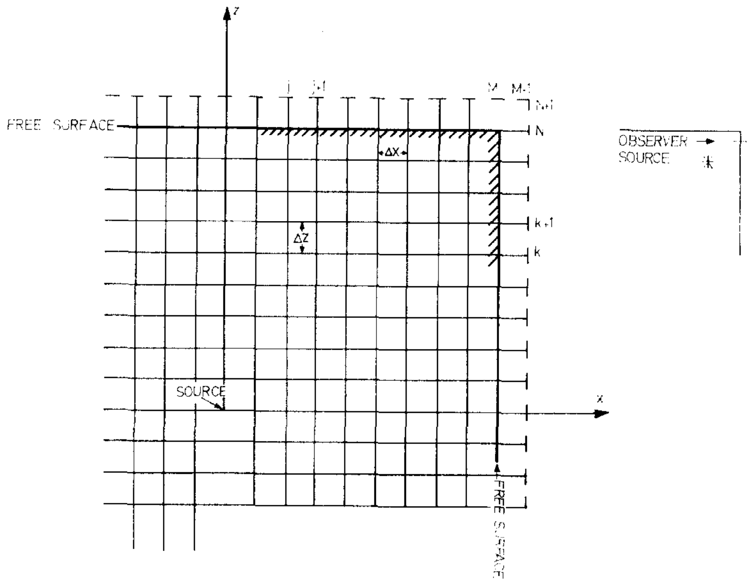

FIG. 1. The grid imposed on the quarter plane.

3. Finite Difference Scheme

A rectangular grid is superimposed on the quarter-plane (see Fig. 1) with increments Δx, Δz in the x and z directions, respectively. A further discretization is assumed in time. $x = j\,\Delta x$, $z = k\,\Delta z$, $t = p\,\Delta t$; where j, k, and p are integers. Let γ denote the ratio of the grid increment Δz to Δx.

$$\gamma = \Delta z/\Delta x.$$

For numerical experiments it is convenient to choose a square grid $\Delta x = \Delta z = h$.

The horizontal and vertical displacements of the grid point (j, k) at time level p is denoted by $(U^p_{j,k}, W^p_{j,k})$.

The time increment Δt is chosen in a way that von Neumann criterion for stability is fulfilled.

$$\begin{aligned} \Delta t &\leqslant \frac{\Delta x}{(\alpha^2 + \beta^2\gamma^2)^{1/2}} \qquad \gamma \leqslant 1 \\ &\leqslant \frac{\Delta z}{(\alpha^2 + \beta^2/\gamma^2)^{1/2}} \qquad \gamma \geqslant 1 \end{aligned} \tag{6}$$

This stability condition was determined by Alterman and Loewenthal [3] and guarantees stability of the finite difference scheme for Eq. (1) in infinite domains. An appropriate choice of the overall dimensions of the grid can guarantee that no artificial reflection will contaminate the results up to a certain desired time.

The finite difference formulas for calculating the displacements of inner points are derived by replacing the derivatives of the equation of motion (1) with central differences. These formulas have been given by Alterman and Rotenberg [2] and many others and will not be repeated here. Along the stress-free furface a special treatment is needed.

Four different approximations for boundary conditions have been compared and their influence on the stability and accuracy of the results have been investigated. The same approximations were studied in [8] in a model of a half-plane. In a quarter-plane the interaction between the horizontal and the vertical boundary as well as the treatment of the grid points near the corner may cause additional problems that are described later.

Let $z = N\Delta z$ be the horizontal stress-free surface. The first two formulas according to Alterman and Karal [1] and Alterman and Rotenberg [2] need the aid of fictitious points located on $z = (N + 1)\,\Delta z$.

1. *Central Approximation*

$$\begin{aligned} U^p_{j,N+1} &= U^p_{j,N-1} - \gamma(W^p_{j+1,N} - W^p_{j-1,N}), \\ W^p_{j,N+1} &= W^p_{j,N-1} - \gamma(1 - 2\delta)(U^p_{j+1,N} - U^p_{j-1,N}). \end{aligned} \tag{7}$$

2. *One-Sided Approximation*

$$\begin{aligned} U^p_{j,N+1} &= U^p_{j,N} - \gamma/2(W^p_{j+1,N} - W^p_{j-1,N}), \\ W^p_{j,N+1} &= W^p_{j,N} - \gamma/2(1 - 2\delta)(U^p_{j+1,N} - U^p_{j-1,N}). \end{aligned} \tag{8}$$

3. *Composed Approximation*

$$\begin{aligned} U^{p+1}_{j,N} &= -U^{p-1}_{j,N} + 2\left[1 - \beta^2\left(\frac{\Delta t}{\Delta z}\right)^2 - \beta^2\left(\frac{\Delta t}{\Delta x}\right)^2(3 - 2\delta)\right] U^p_{j,N} \\ &\quad + 2\left(\frac{\Delta t}{\Delta z}\right)^2 \beta^2 U^p_{j,N-1} + \beta^2\left(\frac{\Delta t}{\Delta x}\right)^2(3 - 2\delta)(U^p_{j+1,N} + U^p_{j-1,N}) \\ &\quad - \beta^2\frac{\Delta t^2}{\Delta x\,\Delta z}(W^p_{j+1,N} - W^p_{j-1,N}), \\ W^{p+1}_{j,N} &= -W^{p-1}_{j,N} + 2\left[1 - \alpha^2\left(\frac{\Delta t}{\Delta z}\right)^2 + \alpha^2\left(\frac{\Delta t}{\Delta x}\right)^2(1 - 2\delta)\right] W^p_{j,N} \\ &\quad + 2\alpha^2\left(\frac{\Delta t}{\Delta z}\right)^2 W^p_{j,N-1} - \alpha^2(1 - 2\delta)\frac{\Delta t^2}{\Delta x\,\Delta z}(U^p_{j+1,N} - U^p_{j-1,N}) \\ &\quad - \alpha^2(1 - 2\delta)\left(\frac{\Delta t}{\Delta x}\right)^2(W^p_{j+1,N} + W^p_{j-1,N}), \end{aligned} \tag{9}$$

where $\gamma = \Delta z/\Delta x$ and $\delta = \beta^2/\alpha^2$.

4. *New Composed Approximation*

$$
\begin{aligned}
W_{j,N}^{p+1} = {} & -W_{j,N}^{p-1} + 2\left[1 - \alpha^2\left(\frac{\Delta t}{\Delta z}\right)^2 - \beta^2\left(\frac{\Delta t}{\Delta x}\right)^2\right] W_{j,N}^{p} \\
& + 2\alpha^2\left(\frac{\Delta t}{\Delta z}\right)^2 W_{j,N-1}^{p} + \beta^2\left(\frac{\Delta t}{\Delta x}\right)^2 (W_{j+1,N}^{p} + W_{j-1,N}^{p}) \\
& + 0.5\,\frac{\Delta t^2}{\Delta x\,\Delta z}(3\beta^2 - \alpha^2)(U_{j+1,N}^{p} - U_{j-1,N}^{p}) \\
& + 0.5\,\frac{\Delta t^2}{\Delta x\,\Delta z}(\beta^2 - \alpha^2)(U_{j+1,N-1}^{p} - U_{j-1,N-1}^{p})
\end{aligned}
\tag{10}
$$

According to the new composed method the horizontal displacement is calculated in the same way as in the composed approximation (9).

Approximations (9) and (10) were developed by Ilan *et al.* [7, 8]. These approximations of the boundary conditions are of the second-order of accuracy and do not need the aid of fictitious lines. The new composed approximation has been found to be stable for a wide range of materials when applied to half-planes.

Here a vertical stress-free surface is added. Approximations (7)–(10) can be applied on the vertical boundary x-constant after transferring them in the following manner

$$
U \to W; \qquad W \to U; \qquad \gamma \to 1/\gamma; \qquad \partial x \to \partial z; \qquad \partial z \to \partial x.
$$

4. The Corner

Karp and Karal [9] analyzed the stress behavior in the neighborhood of the tip of a wedge. They found that in the steady state the stresses are infinite at the tips of wedges, in which the angle between the surfaces is larger than π and are zero otherwise. For the case of elastic wedges with right angles the displacements as well as the stresses are finite in the corner. But the boundary conditions are not well defined there, as the four requirements (4), (5) cannot be fulfilled simultaneously at the corner.

Alterman and Loewenthal [3] proposed two alternative approaches to the calculation of the displacements at the corner. The first approach is to smooth the corner slightly making the tangent to the boundary at an angle of 45° to both axes. Let us denote the corner by (M, N). Under this assumption the displacements at the corner are calculated by the following formulas:

$$
\begin{aligned}
U_{M,N}^{p} &= \delta(W_{M-1,N}^{p} - W_{M,N-1}^{p}) + (1-\delta)(U_{M-1,N}^{p} - U_{M,N-1}^{p}) + U_{M-1,N-1}^{p}, \\
W_{M,N}^{p} &= \delta(U_{M,N-1}^{p} - U_{M-1,N}^{p}) + (1-\delta)(W_{M,N-1}^{p} - W_{M-1,N}^{p}) + W_{M-1,N-1}^{p},
\end{aligned}
\tag{11}
$$

where $\delta = \beta^2/\alpha^2$.

The second approach is to require that the normal stress components of both the horizontal and the vertical surface are zero, and to ignore the condition $\partial U/\partial z + \partial W/\partial x = 0$.

Under the last assumption the following condition is obtained.

$$\frac{\partial U}{\partial x} = \frac{\partial W}{\partial z} = 0. \tag{12}$$

The finite difference approximation to (12) can be simply

$$\begin{aligned} U_{M,N}^{p} &= U_{M-1,N}^{p}, \\ W_{M,N}^{p} &= W_{M,N-1}^{p}. \end{aligned} \tag{13}$$

Applying (11) or (13) to the corner results in similar solutions. Our experiments show that neither of these approximations causes instability. Generally, mistreatment of a few grid points or even of an individual point may 'explode' the whole solution, as is illustrated in the following section.

5. Accuracy and Stability of Results

Many experiments show that approximations to the boundary conditions in the vicinity of a right corner may have a critical influence on the stability of results.

Alterman and Rotenberg [2] applied the central approximation (7) to their model of a quarter-plane and the results became unstable after a few dozen time steps. The same scheme applied to a half-plane is stable for a wide range of β/α, as was found in [8]. Alterman and Rotenberg [2] reported also that while they relaxed the accuracy and replaced the normal derivatives by one-sided differences (8) they got stable results. When they approximated the tangential derivatives of the boundary conditions by one-sided differences also, the scheme became unstable.

The aim of this section is to point out the cause of instability and to propose a remedy.

The centered approximation (7) represents the free surface boundary conditions in one time level. Concentrating our attention on the aid grid points $(M, N+1)$ $(M+1, N)$ (see Fig. 1) we see that the calculation of the displacements at these points is, therefore, interdependent and needs the solution of a system of four algebraic equations in four unknowns in every time step. Applying the central approximation (7) we have

$$\begin{aligned} U_{M,N+1}^{p} + \gamma W_{M+1,N}^{p} &= U_{M,N-1}^{p} + \gamma W_{M-1,N}^{p}, \\ W_{M,N+1}^{p} + \gamma(1-2\delta)\, U_{M+1,N}^{p} &= W_{M,N-1}^{p} + \gamma(1-2\delta)\, U_{M-1,N}^{p}, \\ W_{M+1,N}^{p} + (1/\gamma)\, U_{M,N+1}^{p} &= W_{M-1,N}^{p} + (1/\gamma)\, U_{M,N-1}^{p}, \\ U_{M+1,N}^{p} + (1/\gamma)(1-2\delta)\, W_{M,N+1}^{p} &= U_{M-1,N}^{p} + (1/\gamma)(1-2\delta)\, W_{M,N-1}^{p}. \end{aligned} \tag{14}$$

The determinant of (14) is zero for all values of grid ratio γ and $\delta = \beta^2/\alpha^2$ so that the equations have no solution and the scheme is unstable.

Using the one-sided approximation (8), a similar system of equations is obtained for the displacements at $(M+1, N)(M, N+1)$. But the determinant of the second system is

$$\det = \tfrac{3}{4}[1 - \tfrac{1}{4}(1 - 2\delta)^2]. \tag{15}$$

Determinant (15) is zero for $\delta = \frac{3}{2}$, $\delta = -\frac{1}{2}$. These values are outside the range of a real δ which is, $0 < \delta = (\beta/\alpha)^2 \leqslant \frac{1}{2}$.

Therefore, the algebraic system of equations has a unique solution for this case,

$$\begin{aligned}
U^p_{M+1,N} &= \frac{1}{1-(1/4)(1-2\delta)^2}\Big\{U^p_{M,N} + \frac{1}{2\gamma}(1-2\delta)(W^p_{M,N-1} - W^p_{M,N}) \\
&\quad - \frac{1}{4}(1-2\delta)^2\, U^p_{M-1,N}\Big\}, \\
W^p_{M+1,N} &= \frac{4}{3}\Big\{W^p_{M,N} + \frac{1}{2\gamma}(U^p_{M,N-1} - U^p_{M,N}) - \frac{1}{4}W^p_{M-1,N}\Big\}, \\
U^p_{M,N+1} &= \frac{4}{3}\Big\{U^p_{M,N} + \frac{\gamma}{2}(W^p_{M-1,N} - W^p_{M,N}) - \frac{1}{4}U^p_{M,N-1}\Big\}, \\
W^p_{M,N+1} &= \frac{1}{1-(1/4)(1-2\delta)^2}\Big\{W^p_{M,N} + \frac{\gamma}{2}(1-2\delta)(U^p_{M-1,N} - U^p_{M,N}) \\
&\quad - \frac{1}{4}(1-2\delta)^2\, W^p_{M,N-1}\Big\}.
\end{aligned} \tag{16}$$

Applying one-sided approximations to both x and z derivatives of Eqs. (4) and (5), a system of equations for $(M+1, N)(M, N+1)$ is obtained with a zero determinant for every grid and every medium.

This analysis explains the cases of instability reported by Alterman and Rotenberg [2]. The following numerical experiment has shown that the mistreatment of points $(M+1, N)(M, N+1)$ was the only reason for instability. Figure 2a shows the components of displacements versus $t\alpha/h$ on the surface of a half-plane containing an impulsive point source. The central approximation to the boundary conditions (7) was applied. The same approximation was applied to a quarter-plane in Fig. 2b to obtain an unstable result. In Fig. 2c the central approximation was applied to all the surface grid points except points $(M+1, N)(M, N+1)$ where Eqs. (16) were used. Figure 2c demonstrates that a better understanding of the causes of instability in a quarter-plane enables us to use the more accurate central approximation to the boundary conditions and yet to maintain the stability of the scheme. Repeating this experiment for the two different approximations for the corners, (11) and (13), gives similar results.

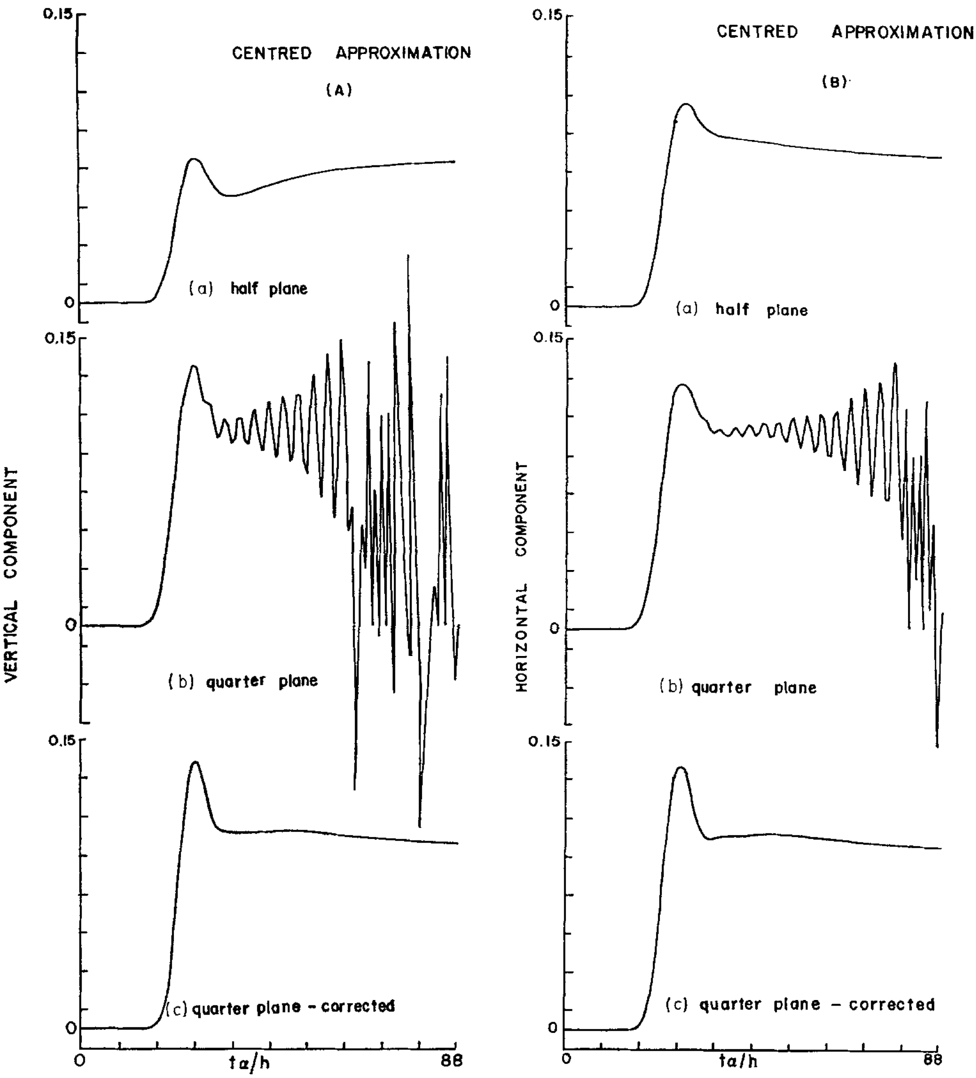

FIG. 2. The components of displacements as functions of $t\alpha/h$, as obtained by the central approximation to the boundary conditions (a) on a half plane (b) on a quarter plane (c) on a quarter plane after applying the corrected scheme to the points near the corner. The source is located at a depth of $10h$, at a horizontal distance of $10h$ from corner, here $\beta/\alpha = 0.58$.

The composed and the new composed approximations, (9) and (10), like the main scheme for inner points are three-level schemes. The displacements in the time step $p + 1$ are dependent on the displacements of time steps p and $p - 1$. Therefore, no implicit system of equations exists for these cases.

The following numerical experiment has been performed in order to evaluate the accuracy of results and to study the range of applicability of each scheme. The four approximations to the boundary conditions (including the correction to the central approximation) were applied to the same model and the results were compared. In order to emphasize the effect of the corner and of the two perpendicular surfaces, the source was situated at a distance of 10h from each surface, and the observers were located in the vicinity of the corner. Figures 3, 4, and 5 show the components of displacements versus the dimensionless time $t\alpha/h$ in quarter-planes with three different ratios of parameters. The observation point was chosen on the surface at a distance of 6h from the corner. At this point the direct pulse P and the converted shear pulse PS can be distinguished. Phase PP apparently coincides with the direct pulse. The arrival times of the different phases were computed according to the geometric ray theory.

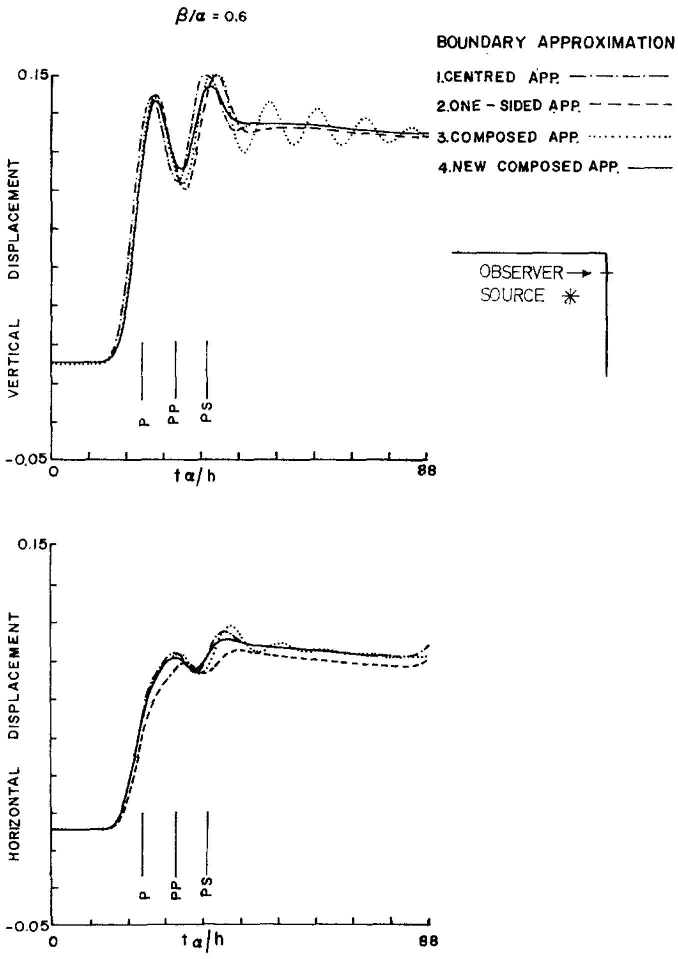

FIG. 3. Displacements versus $t\alpha/h$ (a) vertical (b) horizontal component. Comparison of four approximations to the stress free boundary conditions for a quarter plane. Here $\beta/\alpha = 0.6$; the point source $S_4(r, t)$ is at a depth of $10h$, and a distance of $10h$ from the corner.

Figure 3 shows the displacements in a quarter-plane in which $\beta/\alpha = 0.6$. For this ratio of wave velocities the four approximations to the boundary conditions result in similar solutions. The result of the composed approximation somewhat oscillate about the real solution, and the PS pulse obtained by the one-sided approximation (8) has a phase delay of 8.5 %. When β/α is made smaller than 0.58 the solution of the composed approximation grows out of bounds (Fig. 4 and 5). When β/α is made smaller than 0.35 the result of the one-sided approximation also "explodes" numerically. Experimentally it has been found that the range of stability of each method (7)–(10) in a quarter plane is the same as the range of stability in a half plane.

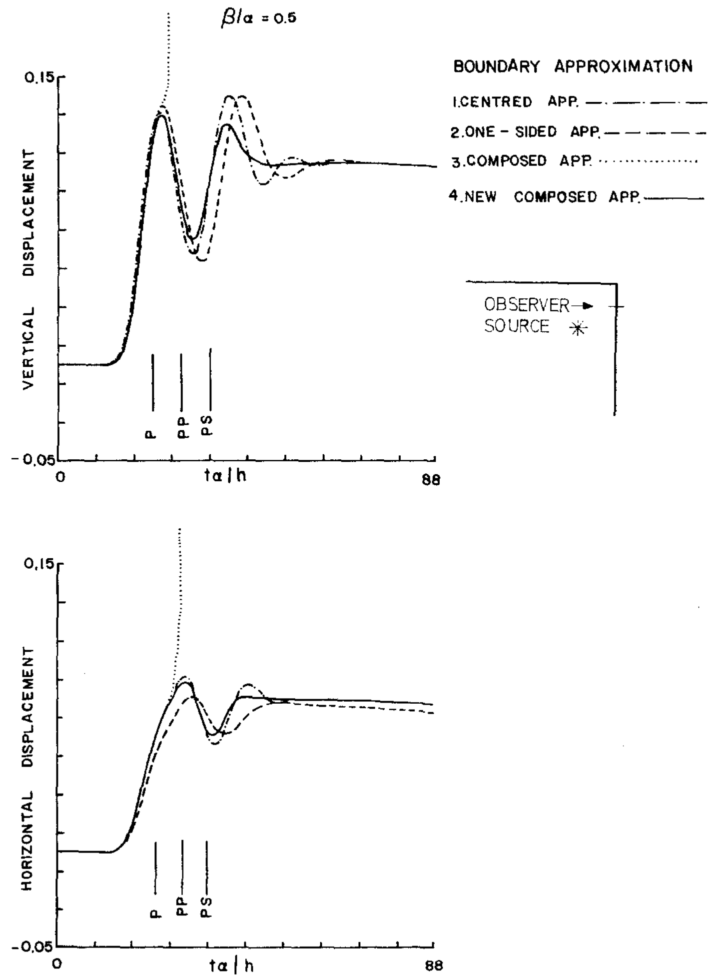

FIG. 4. The components of displacement as functions of $t/\alpha h$ on the surface of a quarter plane. Here $\beta/\alpha = 0.5$ and the other details are the same as in Fig. 3.

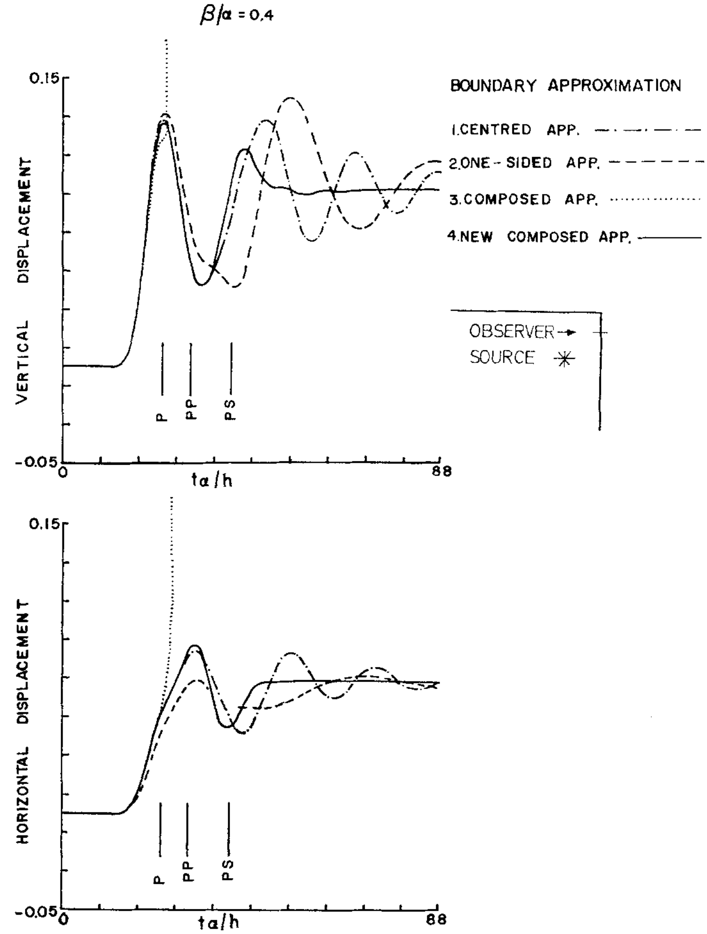

FIG. 5. The components of displacements as functions of $t/\alpha h$ on the surface of a quarter plane. Here $\beta/\alpha = 0.4$ and all the other details are the same as in Fig. 3.

A phase delay is found in solutions of the one-sided approximation (8). The delay increases as β/α decreases as can be seen in Fig. 4 and 5. For small values of β/α the results of the central and the one-sided approximations are contaminated by oscillations about the real solution. The amplitude of the oscillations increases as β/α decreases. For β/α less or equal to 0.5 the phase delay and the oscillations cause unreliable solutions of the central and the one-sided approximation. Only the new composed approximation (10) gives smooth solutions and accurate arrival times for small values of β/α.

6. Analysis of the Local Matrix of Coefficients

The basic idea of finite difference theory is to replace a differential problem by a set of linear algebraic equations. The operator which transfers the solution from one time step to the other can always be represented in a matrix form. This propagating matrix must include the information as to whether the scheme is stable or not. But for an average grid this matrix has huge dimensions and is therefore difficult to analyze. Ilan and Loewenthal [8] showed that it is sufficient to consider a typical small sample of the grid including surface points. Calculating the eigenvalues of the local propagating matrix associated with this sample gave accurate information about the stability of the difference scheme in a half-plane. This method can be easily generalized to a quarter-plane. The small sample of the grid in the last case has to include the corner and a few points on the two perpendicular surfaces. Consider a grid of $m \times n$ points in the vicinity of the corner. The components of displacements of this grid may be arranged in the form of a block vector

$$U^p = \begin{bmatrix} U_1{}^p \\ U_2{}^p \\ U_l{}^p \\ U_m{}^p \end{bmatrix}, \tag{17}$$

where each column $U_l{}^p$ includes the displacements of the l-th row of the grid sample.

$$U_l{}^p = \begin{bmatrix} U^p_{1,l} \\ W^p_{1,l} \\ U^p_{2,l} \\ W^p_{2,l} \\ \vdots \\ U^p_{m,l} \\ W^p_{m,l} \end{bmatrix}. \tag{18}$$

The equation of motion and the boundary conditions on the perpendicular surfaces are represented by the following equation

$$U^{p+1} = QU^p - IU^{p-1}, \tag{19}$$

here I is the identity matrix and Q is a matrix of order 2 mn which is dependent on the parameters of the medium and on the ratio between the grid increments.

Boore [4] showed that for the three level schemes (19) a necessary condition for stability is that no eigenvalue of Q has an absolute value larger than 2.

In the Appendix to [8] an example of a local matrix of propagation Q was given for the composed approximation on a half-plane. This sample included grid points on the free surface and on three artificial boundaries, on which the continuity of the normal displacements was assumed. For a quarter-plane the following modifications were made to the matrix Q. The last two rows were replaced by the formulas representing the right corner [either (11) or (13)], and the last two rows in every submatrix A, B, C were changed to represent the boundary conditions on the vertical stress free surface.

The eigenvalues of Q were calculated numerically for various β/α and for several grid samples, and the stability conditions for infinite plane (6) was always fulfilled. No eigenvalue of Q with absolute value larger than 2 was found in the range of stability of each approximation. On the other hand when β/α was made less than the critical value the spectrum of Q included at least one eigenvalue with absolute value larger than 2. The range of stability was found to be the same for a half- and a quarter-plane and is summarized in Table I.

TABLE I

The Range of Stability in a Quarter Plane

Approximation	Range of Stability
Centerd	$\beta/\alpha > 0.300$
One-sided	$\beta/\alpha > 0.350$
Composed	$\beta/\alpha > 0.575$
New composed	$\beta/\alpha > 0000$

7. Conclusions

Considering several finite difference schemes for a quarter-plane we found that mistreatment of a few grid points near the corner may cause instability of the solution. Analysis of the matrix of propagation in the vicinity of the corner enables us to detect the cause of instability reported by Alterman and Rotenberg [2] and to correct it. Then, the stability of finite difference schemes with four different approximations is found to be the same for the cases of a half-plane and a quarter-plane. In a quarter-plane with β/α less than 0.5 the results of the central and the one-sided approximation are contaminated by oscillations about the real solution and by a phase delay. These inaccuracies increase when β/α decreases. Only the new composed approximation results in smooth solutions with accurate arrival times of the reflected phases in quarter-planes with small values of β/α.

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